SHOCKWAVE FORMATION IN LAVAL NOZZLES

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The computation of flows in Laval nozzles involves great difficulties in the construction of the flow field in the neighborhood of the very narrow transverse section of the channel where the transition is effected from subsonic to supersonic velocities. The first general results on the structure of plane-parallel gas motion in the neighborhood of the sonic curve were obtained by Khristianovich [1]. In particular, a necessary condition was deduced for the analytic continuation of the flow from the subsonic to the supersonic domains in nozzles. The stream singularity in the neighborhood of the channel center is associated with the fact that the tangents to the transition line and the characteristics passing through the axis of symmetry coincide. By relying on the hodograph transformation Frankl' investigated the character of plane-parallel flow near the sonic line in detail [2]. Not only analytic flows but also flows with discontinuities in the first derivatives along the Mach lines passing through the center of the nozzle were examined by Frankl'. Falkovich showed that the results obtained by Frankl' can be simplified considerably if the whole investigation is performed in the plane of the physical variables [3]. Falkovich succeeded in writing the principal term of the solution as a third degree polynomial. The theory of a plane nozzle was also worked out by Lighthill [4], Cherry [5], Ehlers [6], Tricomi [7] and a number of other authors.

The transition through the speed of sound in the neighborhood of the throat of an axisymmetric Laval nozzle was analyzed in [8]. It was shown there that an investigation of the axisymmetric case could be carried out with the same completeness as was achieved in the study of the plane-parallel motions.

Only flows in which shockwaves are absent were examined in all the papers listed. In practical respects, these flows are most interesting. In order to guarantee a flow without shocks in the neighborhood of the center of the nozzle, sufficient conditions were deduced to ensure the potential character of both plane-parallel [7] and axisymmetric motions [8]. However, the reasons leading to the formation of shockwaves in the neighborhood of the nozzle throat were not established. The conditions under which flows are obtained with a compression shock through the center of the channel tangent to the sonic curve and the influence of the induced perturbations on the gas motion in the inlet remained unclear.

First every kind of continuous non-analytic flows is studied here. It turns out that a limit line bearing infinite values of the acceleration appears in such flows under definite conditions. Since gas motion with infinite accelerations is physically without meaning, either a shockwave is formed prior to the appearance of the limit line or the flow as a whole is completely changed. It is established that a shockwave is formed at the center of the nozzle in discontinuous motions and is carried downstream. It is impossible to construct flows with a shockwave arriving at this point; they are destroyed as a result of the perturbations induced by the compression shock. A shockwave is formed only when a limit line appears in the flow first; it is impossible to introduce a shockwave into a flow where there are no infinite accelerations. A compression shock does not disturb the gas motion in the nozzle entrance; the flow beyond it continues to be expanded although more slowly than in continuous flows.

If the throat of the channel is upstream of the point of intersection of the sonic curve and the axis of symmetry, then no shockwaves occur in the gas motion. It is established that the flow also remains shockless when the narrowest section of the nozzle is located further downstream than its center but the distance between them does not exceed a certain limit. An increase in the distance between the throat and the entrance of the channel leads to the formation of shockwaves. Hence, the transition part should not be made too long in constructing nozzles. The origin of a compression shock is least probable in the transonic part of a nozzle whose walls have breaks. Shockwave formation is associated with deceleration in the region behind the characteristic closing the entrance, when it occurs more abruptly than according to a linear law.

Also flows with local supersonic zones adjoining the channel walls and interlocking on its axis are investigated. Such flows were first analyzed by Tomotika and Tamada [9] and Tomotika and Hasimoto [10].

1. Shockless plane-parallel flows. Let us consider the gas motion in the neighborhood of a very narrow transverse section of a nozzle where its character changes from subsonic to supersonic. Let us consider that the particle velocity in that region is near critical in magnitude, and the angles between the direction of the velocity and the axis of the channel are small. The entropy can be taken constant in the whole field of such a flow. The origination of compression shocks in the supersonic part of the flow does not violate the assumptions made since their intensity cannot be large.

Let U and V denote the components of the particle velocity vector along the x and r axes, where the former coincides with the nozzle axis and the latter is selected in a perpendicular direction. The direction of the basic gas motion with critical speed a_* will then be parallel to the x-axis. Let p denote the pressure, s the specific entropy, ρ the density and τ the specific volume; the critical values of the gas parameters will be marked with asterisks. In order to investigate the flow in the transonic velocity range it is convenient to introduce the non-dimensional functions

$$u = 2m_* \frac{U - a_*}{a_*}, \qquad v = 2m_* \frac{V}{a_*} \qquad \left(m_* = \frac{1}{2\rho_*^3 a_*^2}\left(\frac{\partial^2 p}{\partial \tau_*^2}\right)_s, \ \tau = \frac{1}{\rho}\right)$$

which satisfy the system of differential equations [11]

$$-u\,\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + (\nu - 1)\,\frac{v}{r} = 0, \qquad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial x} \tag{1.1}$$

Here $\nu = 1$ for plane-parallel flows and $\nu = 2$ for flows with axial symmetry. We shall also consider the independent variables x and r non-dimensional.

Gas motion without compression shocks cannot be realized through the whole nozzle. In practical respects, nozzles guaranteeing shockless flow are of greatest interest. To elucidate the reasons leading to the formation of shockwaves in the neighborhood of the narrowest section of the channel, let us pose the following Cauchy problem for equations (1.1). At $r = 0$, i.e. on the flow axis of symmetry, let

$$u = A_1 x \ \text{for} \ x < 0, \quad u = A_2 x \ \text{for} \ x > 0, \quad v = 0 \quad (A_1 \geqslant A_2 \geqslant 0) \tag{1.2}$$

Hence, a discontinuity in the derivative $\partial u/\partial x$ is admitted at the point $x = r = 0$. The sonic curve intersects the axis at this point, called the center of the nozzle. The magnitude of the discontinuity in $\partial u/\partial x$ determines the nature of the transition from subsonic to supersonic velocities. Two streamlines symmetric with respect to the x-axis can be taken as the channel walls in a flow constructed as a result of the solution of problem (1.2).

Let us investigate, initially, the plane-parallel motions which contain no compression shocks. Using the stream symmetry, we shall later consider only the upper half-plane of the physical variables xr. The system of equations (1.1) and the initial conditions (1.2) are invariant with respect to the continuous group of similarity transformations

$$x \to \alpha x, \qquad r \to \alpha^{1/2} r, \qquad u \to \alpha u, \qquad v \to \alpha^{3/2} v$$

where α is an arbitrary constant not equal to zero. Hence, we conclude that the desired solution of the Cauchy problem is self-similar; to determine it let us put [3,8]

$$u = r^2 f(\xi), \qquad v = r^3 g(\xi), \qquad \xi = x / r^2 \tag{1.3}$$

For $\nu = 1$ the functions f and g satisfy the following system

$$f \frac{df}{d\xi} - 3g + 2\xi \frac{dg}{d\xi} = 0, \qquad 2\xi \frac{df}{d\xi} + \frac{dg}{d\xi} - 2f = 0$$

Hence, eliminating the function g, we obtain an equation for f

$$(f - 4\xi^2) \frac{d^2 f}{d\xi^2} + \left(\frac{df}{d\xi}\right)^2 + 2\xi \frac{df}{d\xi} - 2f = 0 \tag{1.4}$$

After integrating (1.4), the quantity g is determined by the relation

$$g = \frac{1}{3} [(f - 4\xi^2)\, df/d\xi + 4\xi f] \tag{1.5}$$

Equation (1.4) has a simple particular solution [3], which we shall call "fundamental"

$$f = \frac{1}{2} A^2 + A\xi \tag{1.6}$$

Using the latter equality, let us write the solution

$$u = A_1 x + \frac{1}{2} A_1^2 r^2, \qquad v = A_1^2 x r + \frac{1}{6} A_1^3 r^3 \tag{1.7}$$

of the original system of partial differential equations (1.1), which satisfies the initial conditions (1.2) in a domain located to the left of the singular C_-°-characteristics arriving at the center of the nozzle. The desired solution has a similar form in the domain to the right of the singular C_+°-characteristics issuing from this point

$$u = A_2 x + \frac{1}{2} A_2^2 r^2, \qquad v = A_2^2 x r + \frac{1}{6} A_2^3 r^3 \tag{1.8}$$

For $A_1 = A_2 = A$ (1.7) and (1.8) coincide and also yield the solution of (1.1) in the domain included between the singular $C_{\mp}^{\circ}$-characteristics. This single solution corresponds to the flow in an analytic Laval nozzle. For $A_1 \neq A_2$, with the exception of one special case [8], the solution in the domain between the singular characteristics cannot possibly be represented in such a simple form. In the sequel it will be convenient to consider the quantity A_1 and, therefore, the whole flow in the entrance of the nozzle as unchanged, the values of the constant A_2 will be subject to change.

The sonic line is obtained by equating u in (1.7) to zero

$$x = -\tfrac{1}{2} A_1 r^2$$

The characteristics (Mach lines) are defined by the solutions to the differential equation

$$\left(\frac{dx}{dr}\right)^2 = u = A_{1,2}\, x + \frac{1}{2} A_{1,2}{}^2 r^2$$

In the case under consideration the singular $C_{\mp}^{\circ}$-characteristics, which pass through the center of the nozzle and are tangent to the transition line at this point, are given by the formulas

$$x = -\tfrac{1}{4} A_1 r^2 \quad (C_-^{\circ}\text{ — characteristic}), \qquad x = \tfrac{1}{2} A_2 r^2 \quad (C_+^{\circ}\text{ — characteristic}) \tag{1.9}$$

The neighborhood of the center of the flow is shown in Fig. 1. The C_-°-characteristic is the boundary of regions *1* and *3* and the C_+°-characteristic separates region *3* from region *2*. The solution of equations (1.4) in domain *1* is given by (1.6) with $A = A_1$; in domain *2* by the same equality with $A = A_2$. The functions u and v must remain continuous upon passing through the characteristics while their derivatives may have discontinuities of the first kind. Let us derive two boundary conditions for the integration of (1.4) in domain *3* from the continuity of the function u on the singular $C_{\mp}^{\circ}$-characteristic

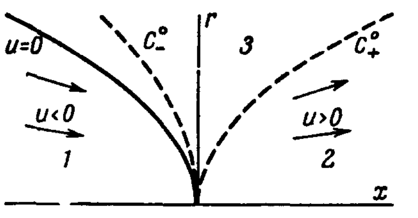

Fig. 1.

$$f = f_1 = {}^1/_4 A_1{}^2 \quad \text{for } \xi = \xi_1 = -{}^1/_4 A_1, \qquad f = f_2 = A_2{}^2 \quad \text{for } \xi = \xi_2 = {}^1/_2 A_2 \tag{1.10}$$

Since $A_1 > 0$ and $A_2 > 0$, then $\xi_1 < 0$ and $\xi_2 > 0$.

In order to simplify the qualitative investigation of nonanalytic gas flows, let us put

$$f = \xi^2 F(\eta), \qquad \frac{dF}{d\eta} = \Psi, \qquad \eta = \ln|\xi| \tag{1.11}$$

Equation (1.4) in the new variables becomes

$$\frac{d\Psi}{dF} = \frac{\Psi^2 + 7\Psi F + 6F^2 - 10\Psi - 6F}{\Psi(4-F)} \tag{1.12}$$

An investigation of the fundamental properties of this equation was carried out in [8]. The general picture of the field of its integral curves is shown in Fig. 2, which shows the character of the singular points, of the curves $\Psi_1{}^*$ and $\Psi_2{}^*$ on which the values of the derivative $d\Psi/dF$ equal zero as well as of the lines $F = 4$ and $\Psi = 0$ where the derivative $d\Psi/dF$ becomes infinite.

By using the $F\Psi$ "phase" plane, let us establish the fundamental properties of the studied flows by drawing initially upon the relation between the values of the constants A_1 and A_2 and the nature of the transition from subsonic to supersonic speeds. Three singular points $A(0,0)$, $C(4, -6)$, $D(4, -12)$ located in the finite part of the $F\Psi$ plane and two infinitely distant singular points E and G, which are reached during motion downward along the lines $\Psi = -2F$ and $\Psi = -3/2F$, respectively, are of interest for the mentioned reasons.

It is easy to show that the point A corresponds to the x-axis; the point C corresponds to the $C_+{}^0$-characteristic and the point D to the $C_-{}^0$-characteristic defined by (1.9); the points E and G correspond to the r-axis. It follows from (1.3) and (1.11) that the ordinate axis corresponds to the sonic line, the half-plane to the right of this axis to the domain of supersonic speeds and the left half-plane to subsonic speeds. If one moves along a certain integral curve in the $F\Psi$ plane then the lines ξ = const will describe a definite region of the physical space. The values of ξ on the considered curve should not have extrema since we will otherwise obtain a multivalued physical plane xr in which the flow is superposed upon itself. The line bearing the extremum value of ξ is the limit line along which the values of the derivatives of the velocity components with respect to the coordinates become infinite. Using (1.11) and (1.12), it is easy to see that passage across the line $F = 4$ is impossible. Only the integral curves passing through the singular points C and D are an exception.

Using (1.11) we find

$$df/d\xi = (2F + \Psi)\,\xi$$

Hence, an equation can be obtained for the integral curve K_1 which will be the image in the $F\Psi$ plane of the fundamental solution (1.6)

$$\Psi = -(1 + 2F \mp \sqrt{1 + 2F}) \tag{1.13}$$

If one moves in the physical space from the subsonic to the supersonic velocity region then the motion along the curve (1.13) will be in the direction shown in Fig. 2 by the arrow. The lower branch of the curve K_1 is the single passage through the point D in the direction E of the integral curve of equation (1.12).

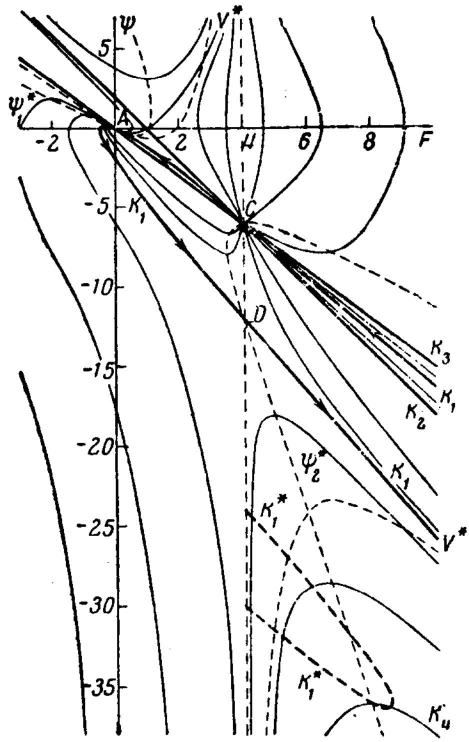

Fig. 2.

Domain *1* in non-analytic flows with weak discontinuities along the singular $C_\mp^0$-characteristics will be shown, as before, by the segment of the curve K_1 located between the points A and D and the gas motion in domain *2* by the segment of the curve K_1 lying between the points C and A since (1.13) depends neither on A_1 nor on A_2. The values of f are continuous on the characteristics and the values of $df/d\xi$ have discontinuities. Hence, it follows that the values of F should also be continuous and, according to (1.10), should equal 4, but the values of Ψ should undergo discontinuities of the first kind. Hence, by moving along the segment of the curve K_1 from the subsonic side and reaching the point D, we obtain the single possibility of realizing flows with weak discontinuities by crossing the shock from the point D to the point C. From the point C it is then possible to move along any integral curve included between two branches of the curve K_1 to the point E and then to return again to the point C along the continuation of this curve.

The linear function

$$\Psi = 2 - 2F \tag{1.14}$$

is an integral of (1.12) corresponding to the line shown in Fig. 2 by K_2. When the gas motion in the domain which is included between the Mach lines passing through the center of the nozzle is mapped by the line (1.14), then this line first proceeds in the customary direction from the point C to the point E and then back, from the point E to the point C. One of the branches of all the remaining integral curves of (1.12) which map the continuous non-analytic flows, is located between the line K_2 and the lower branch of the curve K_1, the other is between this same line and the upper branch of the curve K_1. It is possible to move from the point C to the point E along any branch of the considered integral curves, whereupon we obtain the flow in the domain between the C_-°-characteristic and the r-axis in the physical plane. The flow in the domain between the r-axis and the C_+°-characteristic is mapped by the second branch of the chosen integral curve. In the limiting case the motion will proceed along the curve K_1 drawn in the opposite direction, i.e. from the point D to the point C by a jump, from this point along the curve K_1 to the point E and then along its continuation to the point D and again, by a jump, to the point C.

The study of the field of integral curves of (1.12) permits the establishment of the fundamental properties of flows having weak discontinuities and the elucidation of the reasons leading to the formation of shockwaves in the neighborhood of the nozzle throat.

The flow beyond the C_+°-characteristic is mapped by the segment of the curve K_1 between the points C and A; hence, it follows that discontinuities of the first derivatives of the particle velocity components with respect to the coordinates will not occur on the C_+°-characteristics even if they did on the C_-°-characteristics. The discontinuities in the first derivatives of the velocity components, which occur at the center of a plane nozzle along the C_-°-characteristic, are reflected along the C_+°-characteristic as discontinuities in the second derivatives. The exception in this respect is only the limiting case when the flow between the singular characteristics is mapped on the curve K_1 going in the reverse direction. In the limiting case the discontinuities in the first derivatives of the velocity components with respect to the coordinates are formed both on the C_-°-characteristics and the C_+°-characteristics.

The single integral curve going to the infinitely distant singular point G is the line K_3

$$\Psi = -{}^3/_2 F \tag{1.15}$$

Let us select the segment of the integral curve of (1.12) which is included between the line (1.15) and the upper branch of the curve K_1 as the mapping of the flow between the C_-°-characteristic and the r-axis. Then the flow between the r-axis and the C_+°-characteristics is mapped on the integral curve which is a continuation of the mentioned segment and is located under the lower branch of the curve K_1. As $F \to 4$

the values of Ψ along the considered integral curve decrease and approach infinity in absolute value. Hence, it follows that the $C_+{}^0$-characteristic in the downstream flow constructed in such a manner is simultaneously the limit line since it has infinite values of the derivatives of the velocity components. Under real conditions gas motion with infinitely large accelerations cannot be realized: a shockwave is formed either before the appearance of the limit line or the flow as a whole changes its pattern.

When the flow downstream of the $C_-{}^0$-characteristic in the $F\Psi$ plane is mapped by the curve K_3 then the r-axis is the limit line.

Let us take the integral curve of (1.12) located above the line K_3 as the curve mapping the flow behind the $C_-{}^0$-characteristic. The considered curve intersects the horizontal axis and goes upward toward infinity as $F \to 4$. In the corresponding gas motion, the limit line is to the left of the r-axis; infinite values of the acceleration along it fall at the center of the nozzle.

2. Plane-parallel flows with shockwaves. Let us now examine the formation of shockwaves in the flows with limit lines which we constructed. Shown in Fig. 3 is the neighborhood of the nozzle center at which a compression shock (the curve S) appears; this latter will be more curved than the $C_+{}^0$-characteristic, wherefore only one $C_-{}^0$-characteristic will pass through the nozzle center in the discontinuous gas motion.

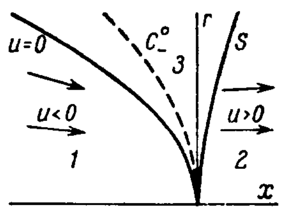

Fig. 3.

The equation of the shock polar for transonic flows may be represented approximately as [12]

$$2\,(v_2 - v_3)^2 = (u_2 - u_3)^2\,(u_2 + u_3) \qquad (2.1)$$

Here the values of the subscripts on the functions u and v denote the numbers of the domains in Fig. 3 in which they are evaluated. Equation (2.1) is a supplementary boundary condition which must be satisfied in constructing discontinuous solutions of the Cauchy problem (1.2). The continuity of the projection V_τ of the velocity vector along the tangent to the shock S is the second boundary condition which must be satisfied in passing through the shock front. Let γ denote the angle between the direction of the velocity and the normal to S. Then

$$V_\tau = U \sin\gamma + V \cos\gamma$$

In gas motions where the particle velocity deviates only insignificantly in magnitude from the critical, the angle γ is small. Hence, it is possible to set $\sin\gamma = dx_2/dr$; $\cos\gamma = 1$ and to consider the shape of the compression shock to be given by the formula $x_2 = x_2(r)$. The continuity condition of the tangential component V_τ of the velocity vector is written as [12]

$$u_2 \frac{dx_2}{dr} + v_2 = u_3 \frac{dx_2}{dr} + v_3 \tag{2.2}$$

Equations (2.1) and (2.2) are also invariant with respect to the above-mentioned group of similarity transformations. Hence, the desired discontinuous solution of the system of equations (1.1) will be self-similar as before and will have the form (1.3), where the solution in domain *1* is given by (1.7) and in domain *2* by (1.8). The parabola

$$x = \xi_2 r^2 \tag{2.3}$$

where the value of the constant ξ_2 depends on the magnitude of the discontinuity of the derivative $\partial u/\partial x$ at the center of nozzle and is subject to determination, is the equation of the shock front.

Condition (2.2) is written as

$$g_2 - g_3 = -2\xi_2 (f_2 - f_3)$$

Using the latter equality, let us simplify the equation of the polar (2.1)

$$f_2 + f_3 = 8\xi_2^2 \tag{2.4}$$

after which let us transform this equality itself by using (1.5) to

$$(f_2 - 4\xi_2^2)\frac{df_2}{d\xi} + 10\xi_2 f_2 = (f_3 - 4\xi_2^2)\frac{df_3}{d\xi} + 10\xi_2 f_3 \tag{2.5}$$

The obtained boundary conditions (2.4) and (2.5) must be satisfied on the right end of the interval when integrating (1.4) in domain *3*. The boundary condition on the left end is given, as before, by the first of formulas (1.10).

To investigate the flows with shockwaves qualitatively, let us again use the $F\Psi$ plane, where at the beginning we shall, as before, dispense with the relation between the character of the transition through sound speed and the values of the constants A_1 and A_2. Equation (2.4) in the $F\Psi$ plane becomes

$$F_2 + F_3 = 8 \tag{2.6}$$

Let us find the relation between the quantities Ψ_2 and Ψ_3 from condition (2.5):

$$\Psi_2 + \Psi_3 = -36 \tag{2.7}$$

Since the flow in domain 2 behind the shockwave is described by (1.8) and, under the assumption made in formulating the Cauchy problem (1.2), its velocity increases; as before, it is mapped by the segment of the K_1 curve between the points C and A. Actually, because of the equalities (1.6) and (1.11)

$$A_2 = (-1 \pm \sqrt{1 + 2F_2})\,\xi_2 \tag{2.8}$$

Hence, it follows that $A_2 \geqslant 0$ for $\xi_2 > 0$ only when $F_2 \geqslant 0$ and the upper sign should be selected in front of the square root. The inequality $\xi_2 > 0$, as will be shown below, is always satisfied. Hence, the quantities F_2 and Ψ_2 are connected by means of relation (1.13) where the upper sign is also taken in front of the root. By using (2.6) and (2.7) it is easy to obtain the connection between the quantities F_3 and Ψ_3 which characterize the state of the gas ahead of the shockwave

$$\Psi_3 = -(19 + 2F_3 + \sqrt{17 - 2F_3}) \tag{2.9}$$

It is necessary to add an inequality expressing the fact that the shock compression process is irreversible. In the considered approximation the entropy in the whole stream is constant, it does not even change when passing from domain 3 to domain 2. The condition characterizing the irreversibility of the shock compression is conveniently taken as

$$u_3 \geqslant u_2$$

since the particle velocity behind the shock front cannot exceed the velocity ahead of it by virtue of the Zemplen theorem. Hence, it follows that

$$F_3 \geqslant F_2$$

Taking account of (2.6), we find that the values of F_3 cannot be less than 4.

In Fig. 2 the equality (2.9) determines the segment of the lower branch of the curve $K_1{}^*$ located in the interval $4 < F \leqslant 8$. Upon approaching the left end of this segment flows are obtained with vanishingly weak shockwaves; the right end corresponds to a flow in which the velocity behind the shockwave equals the critical velocity in magnitude but is parallel to the x-axis in direction. Only part of the integral curves of (1.12) intersect the curve $K_1{}^*$. This latter is a continuation of the integral curves adjoining the upper branch of the curve K_1 and included between it and the line K_3. Continuations of all the remaining integral curves issuing from the point C and located below the line K_3 do not intersect the curve $K_1{}^*$. Hence, only part of the flows in which the limit line is the $C_+{}^{\circ}$-characteristic extending downstream can be realized if shockwaves are introduced. The class of discontinuous flows can be broadened but then the velocity in domain 2 behind the shockwave will diminish along the direction toward the exhaust section of the nozzle. These flows will be examined later. When the limit line in the flow is located to the left of the r-axis or coincides with it then the gas motion cannot be continued successfully through the compression shock. It is impossible to realize such flows in practice.

It should be noted that the compression shock is generated at the nozzle center in all the discontinuous gas flows and extends downstream. This follows directly from the fact that it is possible to intersect the curve $K_1{}^*$, which is the image of the state of the gas ahead of the shock front, only by bypassing the infinitely distant point E for motions along the integral curves in the $F\Psi$ plane. As has been mentioned in the previous section, this corresponds to the r-axis. Actually, the inequality $\xi_2 > 0$ always holds. It is impossible to construct flows with shockwaves arriving at the center of the nozzle. If a shockwave were to be generated in the domain to the left of the r-axis, then the gas motion would be transformed as a result of the induced perturbations.

The $C_-{}^{\circ}$-characteristic at the center of the nozzle is the bearer of the discontinuities of the first derivatives of the velocity vector components with respect to the coordinates both in continuous non-analytic flows and in flows with shockwaves. Both kinds of gas motions are mapped in the $F\Psi$ plane by curves which issue from the point C; only a shock from the point D can hit at this point. Hence, the discontinuities in the first derivatives of the velocity components can be reflected from the nozzle center both as weak discontinuities and as discontinuities in the functions themselves.

Let us note that the shockwaves originate only in flows having limit lines. It is impossible to keep such flows shockless. At the same time it is impossible to introduce compression shocks in motions in which limit lines have not been formed first. Germain and Gillon [13,14] and Gor'kov and Pitaevskii [15] obtained analogous results in the investigation of the problem of incidence of a weak discontinuity on a transition line.

3. Dependence of the character of the transition through sonic speed on the magnitude of the discontinuity of the derivative $\partial u/\partial x$ at the center of the nozzle. Let us turn to the elucidation of the character of the integral curves of (1.4), which correspond to the curves considered above in the $F\Psi$ plane.

The integral curves in the ξf plane determine the change in the magnitude of the nondimensional perturbation velocity u as a function of the nozzle length for r = const. To a first approximation the lines r = const may be equated with streamlines, hence, the solutions of (1.4) give the change in the velocity and also the pressure, density and temperature along the streamline in the neighborhood of the nozzle throat. Let us note that the magnitude of the discontinuity in the derivative $\partial u/\partial x$ on the C_-°-characteristic cannot be arbitrary. Actually, by evaluating $df/d\xi$ upon approaching the C_-°-characteristic from the right we have

$$\frac{df}{d\xi} = -\frac{1}{2}A_1 \tag{3.1}$$

Hence, denoting the magnitude of the discontinuity in the derivative $\partial u/\partial x$ on the C_-°-characteristic by $[\partial u/\partial x]_1$, we obtain

$$\left[\frac{\partial u}{\partial x}\right]_1 = -\frac{3}{2}A_1 < 0$$

Hence, by prescribing the flow to the left of the C_-°-characteristic, it is possible to obtain both continuous flows and flows with shockwaves to the right of it but the value of the jump $[\partial u/\partial x]_1$ along the boundaries of the flow will be the same in all the non-analytic flows.

Let us examine the integral curve of (1.4) to which corresponds a certain curve in the F plane starting and ending at the point C, i.e. the curve which maps shockless gas motion. Approaching the C_+°-characteristic from right and left, we have

$$\frac{df_2}{d\xi} = A_2 \tag{3.2}$$

and in conformity with the above

$$\left[\frac{\partial u}{\partial x}\right]_2 = 0$$

Equations (1.10), (3.1) and (3.2), which must be satisfied at the ends of the interval when integrating (1.4) in the domain 3 can be given the form

$$f_1 = 4\xi_1^2, \quad \frac{df_1}{d\xi} = 2\xi_1 \quad \text{for } \xi = \xi_1, \qquad f_2 = 4\xi_2^2, \quad \frac{df_2}{d\xi} = 2\xi_2 \quad \text{for } \xi = \xi_2 \tag{3.3}$$

Simultaneous compliance with the last four equations is made possible because the ends of the interval $P(\xi_1, f_1)$ and $Q(\xi_2, f_2)$ are singular points of (1.4) through which a non-denumerable manifold of integral curves with the same slope passes. In fact, evaluating the roots $df^*/d\xi$ of the expression

$$\left(\frac{df}{d\xi}\right)^2 + 2\xi\frac{df}{d\xi} - 8\xi^2 = 0$$

we have

$$\frac{df_{1,2}}{d\xi} = (-1 \pm 3)\,\xi$$

This assertion also follows from the fact that the point C is singular for (1.12).

Using the inequalities $\xi_1 < 0$, $\xi_2 > 0$ and formula (3.3), it is easy to see that $df_1/d\xi < 0$ and $df_2/d\xi > 0$, i.e. the function f is not monotonic in the considered interval when the flow is shockless. In conforming with this, the quantity u, which is proportional to the non-dimensional perturbation velocity, first starts to decrease in shockless flows and then grows in the region between the singular $C_{\mp}^{\circ}$-characteristics along the line r = const.

The integral

$$f = 3B^2 + \xi^2 \tag{3.4}$$

of equation (1.4), where the letter B denotes an arbitrary constant, corresponds to the solution of (1.14) in the ξf plane. Let us evaluate the function g by using formula (1.5)

$$g = 6B^2\xi - \tfrac{2}{3}\xi^3$$

Hence

$$u = 3B^2r^2 + \left(\frac{x}{r}\right)^2, \quad v = 6B^2xr - \frac{2}{3}\left(\frac{x}{r}\right)^3 \tag{3.5}$$

The singular $C_{\mp}^{\circ}$-characteristics are located symmetrically with respect to the r-axis in the constructed flow

$$x = -Br^2 \quad (C_-^{\circ}\text{-characteristic}), \qquad x = Br^2 \quad (C_+^{\circ}\text{-characteristic}) \tag{3.6}$$

By satisfying the boundary conditions in the form (1.10), we find

$$B = \tfrac{1}{4}A_1, \qquad A_2 = 2B$$

Hence, it follows that the considered gas motion is realized in the domain included between the singular Mach lines (3.6) when the value of the derivative $\partial u/\partial x$ in the nozzle entrance is twice the value of this derivative in domain 2. As formulas (3.5) show, the analytic continuation of the flow in the domain exterior to the $C_{\mp}^{0}$-characteristics has a Prandtl-Meyer type singularity at the point $x = r = 0$. The direction of the particle velocity in the solution (3.5) is parallel to the channel axis at points of the r-axis and along the parabolas

$$x = \mp 3Br^3$$

located symmetrically to the r-axis behind the Mach lines passing through the center of the nozzle. Among the lines mentioned only the r-axis belongs to the flow in the vicinity of the critical section.

Let us put a minus sign before the B^2 in (3.4) and (3.5); then we obtain the gas flow mapped in Fig. 4. Although it has no direct relation to the problem, it is still interesting as an example of mixed flow whose supersonic part transforms into a centered simple wave at the point $x = r = 0$.

Let us turn now to the investigation of the limiting case of shockless non-analytic flows whose image in the $F\Psi$ plane is the K_1 curve going in the reverse direction. Equation (1.4) has the following solution in domain 3 between the singular Mach lines

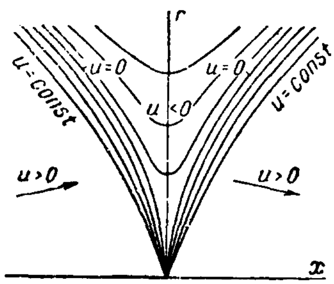

Fig. 4.

$$f = \tfrac{1}{8} A_1{}^2 - \tfrac{1}{2} A_1 \xi \qquad (3.7)$$

which is of the form (1.6) and where the constant is assigned the value $-1/2\ A_1$. The gas flow in this domain is described by the simple formulas (1.7) to (1.8), where it is also necessary to make the mentioned change in the constant. The quantity A_2 is expressed in terms of A_1 thus

$$A_2 = \tfrac{1}{4} A_1 \qquad (3.8)$$

It follows from (3.7) that the derivative $df/d\xi$ is everywhere less than zero in this limiting case of shockless flows with weak discontinuities, i.e. the velocity along the lines r = const in the flow domain bounded by the $C_{\mp}^{0}$-characteristics decreases monotonically.

The flows mapped in the $F\Psi$ plane by the curve K_1 passing in the forward and reverse directions will be among the continuous limiting flows, hence, it is easy to obtain the Frankl' inequalities [2]

$$\frac{1}{4} \leqslant A_2 / A_1 \leqslant 1 \qquad (3.9)$$

which are sufficient to ensure the shockless character of the flow. Let us emphasize the following fact which is established directly from the analysis made of the integral curves in the $F\Psi$ plane. In the domain located downstream of the C_-°-characteristic near the center of the nozzle, the derivative $\partial u/\partial x$ which determine the rate of acceleration of the subsonic flow into supersonic, cannot exceed its value in the entrance. Even in the case when the nozzle has the shape shown in Fig. 5, and the Prandtl-Meyer flow is realized in the vicinity of the break in the wall, the equality

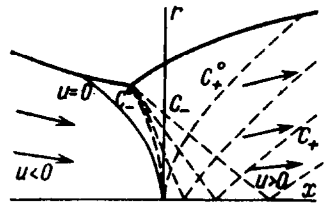

Fig. 5.

$$\left(\frac{\partial u}{\partial x}\right)_{x=-0,\ r=0} = \left(\frac{\partial u}{\partial x}\right)_{x=+0,\ r=0}$$

holds.

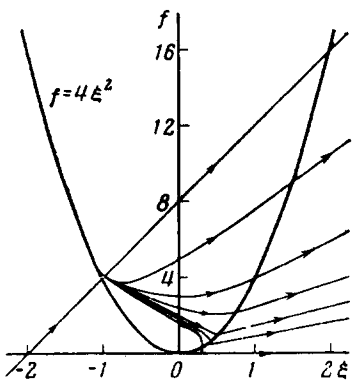

Fig. 6.

The behavior of the integral curves in the ξf plane is shown in Fig. 6. For continuous flows both the boundary points $P(\xi_1,\ f_1)$ and $Q_i(\xi_{2i},\ f_{2i})$ lie on the parabola $f = 4\xi^2$, as follows directly from equation (3.3). The line $f = 1/2\ A_1^{\,2} + A_1\xi$ which corresponds to an analytic flow and the line $f = 1/8\ A_1^{\,2} - 1/2\ A_1\xi$ which corresponds to a flow with discontinuities in the first derivatives on both the singular characteristics are limit lines. All other integral curves in the ξf plane issuing from the point P and describing shockless gas flow are located between them. The derivative $df/d\xi$ on both ends of the considered curves is bounded.

Relation (3.4) yields a curve which is symmetric relative to the r-axis. Above it are the curves to which correspond integral curves of (1.12) in the $F\Psi$ plane with the following branch location. Their branches going from the point C to the infinitely distant point E are included between the line K_2 and the lower branch of the curve K_1; the branches going in the opposite direction from the point E to the point C are between the line K_2 and the upper branch of the curve K_1. Between the curve (3.4) and the straight line (3.7) are the integral curves of equation (1.4), to which correspond the same curves in the $F\Psi$ plane which start and end at the point C but whose branches go in the reverse direction.

The integral curves above the curve (3.4) are obtained for $1/2 < A_2/A_1 < 1$; the integral curves included between the curve (3.4) and the line (3.7) are obtained for $1/4 < A_2/A_1 < 1/2$.

The requirement of continuity of the function f when passing through the singular Mach lines which was satisfied up to now in the construction of shockless flows, guarantees the continuity of the stream velocity component u along the x-axis. For continuity of the quantity v on the $C_{\pm}^{\circ}$-characteristics, it is necessary that the function g should also not have discontinuities. Since $f = 4\xi^2$ along the Mach lines tangent to the sonic curve at the center of the nozzle, it follows from (1.5) that this condition is also satisfied.

Let us now consider the flow of a gas with shockwaves. In the $F\Psi$ plane they are mapped by curves going from the point C to the point E above the upper branch of the curve K_1; their continuations, starting at the point E, go downward toward infinity as $F \to 4$. The corresponding integral curves of (1.4) which describe flows with shockwaves separating domains *3* and *2* are located on Fig. 6 under the line (3.7). A curve which maps the flow with critical values of the gas parameters behind the compression shock is their lower boundary. Let us note that the slope of the considered integral curves in the ξf plane diminishes continuously and becomes infinite in absolute value at the points of intersection with the right branch of the parabola $f = 4\xi^2$. At the same time the slope of the curves going above the line (3.7) has a minimum value at the point P and then increases continuously. Hence, it follows that the reason for the origin of shockwaves in the vicinity of the critical section of the nozzle is the fact that the stream velocity behind the C_-°-characteristic decreases more sharply than according to a linear law.

Since the right branch of the parabola $f = 4\xi^2$ corresponds to a C_+°-characteristic, the state of the gas before the shockwave is mapped by points lying on it; the state behind the shockwave by points below it. In the limiting case when the velocity behind the shockwave coincides with the critical velocity, the ξ-axis corresponds to the flow behind the shockfront in Fig. 6. In this case $A_2 = 0$ and a uniform stream flowing with the speed of sound parallel to the x-axis is realized in the whole domain 2. Hence, we conclude that in the range of values

$$0 < A_2 / A_1 < \tfrac{1}{4} \tag{3.10}$$

a shockwave is formed in the flow on both sides of which the velocity is supersonic. The inequalities (3.10) also guarantee a further increase in the velocity in domain 2.

The integral curve

$$f = d\sqrt{\xi}, \qquad d = \text{const} \tag{3.11}$$

of (1.4), which touches the origin $\xi = f = 0$ and has a derivative infinite in absolute value there, corresponds to the line (1.15) in the $F\Psi$ plane. According to the results obtained in the preceding section, the integral curve describing the flow with critical values of the parameters behind the shockwave is located above the curve (3.11).

The formula

$$f = 4c\xi + 8c^2 + d\sqrt{c+\xi} \tag{3.12}$$

obtained by Fal'kovich [3] and giving the general solution of (1.4) as a function of two arbitrary constants c and d can be used for the exact construction of the integral curves in domain 3. For $c = 1/4\,A$ and $d = 0$ we hence have the fundamental solution (1.6), for $c = 0$ formula (3.12) transforms into (3.11). The singular solution (3.4) can also be deduced from the equation presented by passing to the limit, as will be mentioned below.

Using (3.5) we have

$$g = \frac{1}{3}\left(48c^2\xi + 32c^3 + \frac{1}{2}d^2 + 8cd\sqrt{c+\xi} + 2d\xi\sqrt{c+\xi}\right)$$

Hence, the nondimensional components u and v of the stream velocity in domain 3 approach the limits

$$\begin{aligned} u &= 4cx + 8c^2r^2 + dr\sqrt{x+cr^2} \\ v &= \frac{1}{3}\left[48c^2xr + \left(32c^3 + \frac{1}{2}d^2\right)r^3 + 2d(4cr^2 + x)\sqrt{x+cr^2}\right] \end{aligned} \tag{3.13}$$

The relations (3.13) show that in the entrance and exhaust sections of the nozzle only the fundamental solution (1.6) yields gas motion which is symmetric relative to the x-axis.

Before calculating the values of the constants c and d, let us note that equation (3.12) does not change upon transformation to the new quantities

$$\xi \to \frac{\xi}{4/A_1}, \qquad f \to \frac{f}{16/A_1^2} \tag{3.14}$$

Such a substitution is convenient in that any gas motion in domain 1 is described in the new variables by the formula

$$f = 8 + 4\xi$$

independently of the values of the constant A_1 while all the remaining expressions retain their previous form. Let that substitution be made. Then the point P in Fig. 6 which maps the C_-^0-characteristic will have the coordinates $\xi = -1$, $f = 4$. All the integral curves of (1.4) passing through domain *3* must start at the point P and must have the derivative $df/d\xi = -2$ there. Hence, we find a relation between the constants c and d

$$d = -8\left(c + \tfrac{1}{2}\right)\sqrt{c-1} \tag{3.15}$$

Let us analyze shockless flow with weak discontinuities along the singular Mach lines. The integral curve mapping it should have the bounded derivative $df/d\xi = 2\xi_2$ at the point $Q(\xi_2, 4\xi_2^2)$. This requirement determines the constant c

$$c = \frac{\xi_2^2 - \xi_2 + 1}{3(\xi_2 - 1)}$$

Using the last two formulas, we find

$$f = \frac{4}{3}\frac{\xi_2^2 - \xi_2 + 1}{\xi_2 - 1}\xi + \frac{8}{9}\frac{(\xi_2^2 - \xi_2 + 1)^2}{(\xi_2 - 1)^2} + \\ + \frac{4}{9}\frac{(2\xi_2^2 + \xi_2 - 1)(\xi_2 - 2)}{(\xi_2 - 1)^2}\sqrt{3(\xi_2 - 1)\xi + \xi_2^2 - \xi_2 + 1} \tag{3.16}$$

Let the value of the coordinate ξ_2 tend to 1. Hence, $c \to \pm\infty$. Performing the passage to the limit as $\xi_2 \to 1$, we obtain the integral (3.4) from the solution (3.16). The solution (3.16) satisfies all the boundary conditions at the points P and Q only if

$$\tfrac{1}{2} \leqslant \xi_2 \leqslant 2$$

The obtained relation is easily transformed to the form (3.9) in which the condition guaranteeing the shockless character of the flow was written.

Now, let (3.12) and (3.13) describe the flow with a compression shock issuing from the center of the nozzle. The relation (3.15) between the corresponding constants will hold as before. The curve mapping the discontinuous flow in the ξf plane intersects the parabola $f = 4\xi^2$ when $\xi = \xi_*$, whereupon according to the preceding

$$0 < \xi_* < \tfrac{1}{2}$$

The derivative $df/d\xi$ becomes infinite in absolute value at the point $Q(\xi_*, 4\xi_*^2)$. The integral curve satisfying the imposed requirements can be determined as

$$f = -4\xi_*\xi + 8\xi_*^2 + 8\left(\tfrac{1}{2} - \xi_*\right)\sqrt{(\xi_* + 1)(\xi_* - \xi)}$$

The coordinate ξ_2 of the compression shock in (2.3) and the quantities f_2 and f_3 are found from the boundary conditions (2.4) and (2.5) but the latter are too awkward and have no explicit solutions. It is more convenient again to turn to the $F\Psi$ plane to calculate them, where the general solution of (3.12), (3.13) can be represented as

$$F = -4z + 8z^2 + ez^{3/2}(z-1)^{1/2}$$
$$\Psi = -\left[-4z + 16z^2 + 2ez^{3/2}(z-1)^{1/2} + \tfrac{1}{2}\,ez^{3/2}(z-1)^{-1/2}\right] \tag{3.17}$$

Here

$$e = 4\,\frac{1 - \xi_* - 2\xi_*^2}{\xi_*^{3/2}(1+\xi_*)^{1/2}}, \qquad z = \frac{\xi_*}{\xi} \tag{3.18}$$

Only those branches of the curves (3.17) which are obtained for $z \geqslant 1$ are of interest for the formulated problems. Points of their intersection with the curve (2.9) yield the values F_3 and Ψ_3 corresponding to the state of the gas ahead of the shock front. The values F_2 and Ψ_2 of the quantities F and Ψ behind the compression shock are obtained from (2.6) and (2.7). Let z_2 denote the magnitude of the parameter z for which the intersection of the curves (2.9) and (3.17) is achieved. The coordinate ξ_2 of the compression shock in (2.3) is found from the equality (3.18) if we set $z = z_2$. The constant A_2 which expresses the rate of stream broadening in domain 2 is calculated from relation (2.8). Finally, using (1.11) and then (1.5), it is possible to determine the values of the functions f and g on both sides of the shockwave.

Figure 7 shows the dependence of ξ_2 on the constant A_2 constructed by taking account of the substitution (3.14). In the range of values $1 \leqslant A_2 \leqslant 4$ the C_+°-characteristic is the line ξ_2 = const. In this case the relation between ξ_2 and A_2 is linear; it is calculated by means of the last of formulas (1.10). In the range $0 \leqslant A_2 < 1$ the compression shock is the parabola ξ_2 = const. As is seen from Fig. 7, its position changes slightly with changes in A_2 as $A_2 \to 0$. The shape of the shock front behind which the gas parameters achieve critical values is given by the equality

$$x = 0.075\ A_1 r^2$$

When $A_2 \to 1$ and the compression shock intensity is small, the dependence of ξ_2 on A_2 is almost linear. For $A_2 = 1$ the slope of the curve determining the position of the shock front agrees with the slope of the line $\xi_2 = 1/2\ A_2$ which gives the position of the C_+°-characteristic. In conformance with (2.4) and (1.11) we have

$$f_3 - f_2 = \xi_2^2(8 - 2F_2)$$

The described relation determines the compression shock intensity, i.e. the discontinuity in the magnitudes of the velocity, pressure, density and temperature during passage through the shock front. The discontinuity in the magnitude of the slope of the velocity vector relative to the nozzle axis is given by the equations

$$g_3 - g_2 = -2\xi_2^3(8 - 2F_2)$$

Figure 8 shows the dependence of $f_3 - f_2$ on the constant A_2 constructed also by taking account of the substitution (3.14). As is seen from Fig. 8, at the beginning (for $1 > A_2 > 0.9$) the compression shock

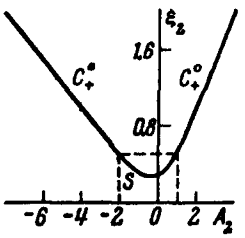

Fig. 7.

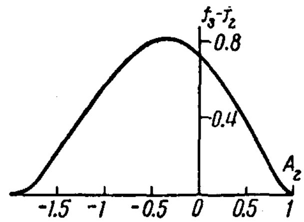

Fig. 8.

intensity grows slowly as A_2 diminishes. For $0.1 < A_2 < 0$, the relation between $f_3 - f_2$ and A_2 seems almost linear. Using the data presented in Figs. 7 and 8, it is easy to obtain the difference $g_3 - g_2$ as a function of A_2.

Up to now flows have been subjected to investigation whose character in the vicinity of the nozzle throat changed from subsonic to supersonic. The analyzed type of gas motion is realized when the difference in the pressures at the nozzle entrance and exit is sufficiently large. If the pressure at the entrance does not radically exceed the pressure at the exhaust, then the flow will be subsonic on both sides of the nozzle throat but it may contain supersonic regions adjoining the walls in the vicinity of the critical section. As the pressure diminishes at the exit, the dimensions of the local supersonic zones increase and, finally, they merge on the channel axis. Such a flow is limiting in the sense that with a further reduction of the pressure in the exhaust the nature of the flow changes and the velocity field behind the critical section becomes supersonic.

4. Flows with local supersonic zones. Let us analyze the limiting gas motion which occurs when local supersonic zones adjoining the nozzle walls are joined on the axis. To do this, let us again analyze the Cauchy problem (1.2) but let us consider that $A_1 > 0$ and $A_2 < 0$. The desired solution in domain *1* will be given by (1.7), as before, and in domain 2 by (1.8), from which it follows that the corresponding flow is subsonic on both sides of the channel throat. The transition line bounding the local supersonic region in shockless flows will be

$$x = -\frac{1}{2}A_1r^2, \qquad x = -\frac{1}{2}A_2r^2$$

While the first of these lines, remaining unchanged, is concave toward the free stream the second is concave downstream. As we shall see later, a compression shock may close a local supersonic zone at the rear in discontinuous flows.

The singular C_-°-characteristic passing through the center of the nozzle is determined by the first of formulas (1.9), the C_+°-characteristic is written as

$$x = -\frac{1}{4}A_2r^2 \tag{4.1}$$

The results of Section 2 show that the C_+°-characteristic exists only in a flow devoid of shockwaves; its structure in the vicinity of the nozzle center is shown in Fig. 9.

The first of the boundary conditions (1.10) remains unchanged when integrating (1.4) in domain 3. If the flow is shockless, then according to (4.1), the second condition of (1.10) must be replaced by

$$f = f_2 = \frac{1}{4}A_2^2 \quad \text{for } \xi = \xi_2 = -{}^1/_4A_2 \tag{4.2}$$

Since $A_2 < 0$, then

$$\xi_2 > 0$$

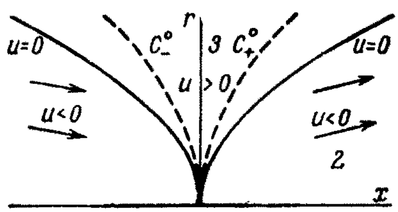

Fig. 9.

If there is a compression shock in the flow, then the boundary condition which must be satisfied on the right end of the interval when integrating (1.4) in domain 3 has the form, (2.4), (2.5), as before.

Let us again use the phase plane $F\Psi$, where as usual we will at the beginning dispense with the relation between the character of the flow in the vicinity of the nozzle center and the values of the constants A_1 and A_2. Since the flow in domain 1 remains unchanged it is mapped in Fig. 2 by the segment of the curve K_1 located between the points A and D and having the previous direction of traversal. According to equation (2.8), $A_2 < 0$ for $\xi_2 > 0$ either when $F_2 < 0$ and the upper sign is selected in front of the root or when the lower sign is taken in front of the root for arbitrary admissible values of F_2. Reasoning, analogous to that in Section 2 shows that the values of ξ_2 remain positive for all the considered flows. Hence, the domain 2 of gas motion with local supersonic zones being closed on the nozzle axis is mapped by a segment of the curve K_1 which is also located between the points A and D but has the opposite direction of traversal, i.e. from D to A.

Let us consider flows devoid of shock fronts. The simplest is a flow which is mapped by the curve K_1 passing in a straight line from the point A to the point C with a subsequent jump from the point C to the point D. The constructed flow agrees in domains *1* and *3* with the motion of a gas passing through the speed of sound in an analytic Laval nozzle. The C_-°-characteristic therein bears no singularities; the first derivatives of the velocity components with respect to the coordinates under discontinuities along the C_+°-characteristics [9]. Satisfying the boundary condition (4.2), we obtain the value of the derivative $\partial u/\partial x$ in domain *2*

$$A_2 = -2A_1 \tag{4.3}$$

This is the least value of A_2 for a given value of A_1. In conformance with (4.3) we obtain

$$[\partial u / \partial x]_2 = -3A_1$$

In all the remaining flows of the considered type, the C_-°-characteristic arriving at the nozzle center is the bearer of discontinuities in the first derivatives of the velocity components. The gas motion in domain *3* is mapped in the $F\Psi$ plane by integral curves starting and terminating at the point C. Hence the parameters of the medium coincide in the whole space to the left of the C_+°-characteristic starting from the nozzle center in flows with local supersonic zones and in corresponding flows with the passage through the speed of sound. The equality (3.1) remains valid with passage through the C_-°-characteristic.

Upon continuing the flow from domain *3* into domain *2* it is necessary to perform a jump from the point C to the point D in the $F\Psi$ plane. The C_+°-characteristic in gas motions with local supersonic zones also bears discontinuities in the first derivatives of the velocity components with respect to the coordinates. By using the last of equalities (3.3), we find the magnitude of this discontinuity

$$[\partial u / \partial x]_2 = {}^3/_2 A_2 < 0$$

Only in the flow whose mapping in the $F\Psi$ plane is the curve K_1 going in the reverse direction is the C_+°-characteristic devoid of any singularity. Actually, in this case the curve K_1 passes from the point A to the point D with a subsequent jump to the point C; from the point C the motion along the curve K_1 proceeds through the infinitely distant point E to the point D and further, through this point again to the point A. The flow in domains *3* and *2* is described by (3.7), from which it follows that the greatest value of A_2 guaranteeing shockless flow with supersonic zones closed on the channel axis is

$$A_2 = -0.5A_1 \tag{4.4}$$

Equation (4.4) is similar to (3.8).

Flows mapped in the $F\Psi$ plane by the curve K_1 traveling in the forward and reverse directions are, again, the limiting flows among the continuous flows. Hence, we find that the range of values of A_2 corresponding to continuous flows is determined by the inequalities

$$-2 \leqslant A_2 / A_1 \leqslant -0.5 \tag{4.5}$$

which supplement Frankl's inequality (3.9) referring to flows with the passage through the speed of sound.

A flow symmetric not only to the channel axis x but also to the r-axis corresponds to the solution (3.4), (3.5). Evidently the value $A_2 = -A_1$ corresponds thereto.

Let us turn to a study of flows with local supersonic zones which contain shockwaves. As in the continuous flows considered above, in the majority of flows with shockwaves the gas parameters in domains *1* and *3* agree with the gas parameters in the corresponding motions where the transition from subsonic to supersonic velocities is performed. The very same curves which emerge from the point C above the upper branch of the curve K_1 and whose continuations, starting with the point E, lie under the lower branch of the curve K_1 are their mapping in the $F\Psi$ plane. However, the jump from the considered integral curves occurs not on the segment CA of the curve K_1 but on the segment DA. Hence, the state of the gas ahead of the shock front is mapped either by points of the upper branch

$$\Psi_3 = -(19 + 2F_3 - \sqrt{17 - 2F_3}) \tag{4.6}$$

of the curve $K_1{}^*$ where $4 < F_3 \leqslant 8.5$ or by points of the lower branch (2.9) of the curve $K_1{}^*$, where $F_s < F_3 \leqslant 8.5$ (Fig. 2). The value F_s of the quantity F is obtained when the curve $K_1{}^*$ intersects the integral curve of (1.12) which goes through the point (8, −36) in the $F\Psi$ plane and maps the gas flow with critical values of the parameters in domain *2*. Let us designate this curve as K_4. The compression shock in flows with local supersonic zones occurs earlier than in the corresponding motions where a passage from subsonic velocities in domain *1* to supersonic velocities in domain *2* is realized.

Integral curves of equation (1.12) which map the flows with a subsonic velocity field in domain *2* behind the shockwave pass below the curve K_4 in the $F\Psi$ plane. A flow without transition through the speed of sound in the vicinity of the critical channel section is not obtained for any intersection of these curves with the lower boundary (2.9) of the curve K_1*.

Flows with shockwaves are given in the ξf plane by curves which go below the line (3.7); their properties were described in the preceding section. The coordinate ξ_2 of the compression shock is found as a result of the joint solution of (3.17) and (4.6) when $4 < F_3 \leqslant 8$; of (3.17), (2.9) and (3.17), (4.6) when $8 < F_3 \leqslant 8.5$. The range of values of A_2 corresponding to discontinuous motions is defined by the inequalities

$$-0.5 < A_2 / A_1 < 0 \tag{4.7}$$

The results of computations showing the dependence of ξ_2 on A_2 for $A_2 < 0$ are shown in the left half of Fig. 7. As before, the substitution (3.14) is taken into account. In the range $-8 \leqslant A_2 \leqslant -2$, the C_+°-characteristic is the parabola ξ_2 = const and the relation between ξ_2 and A_2 is linear, and given by the formula $\xi_2 = -0.25 A_2$. In the range of values $-2 < A_2 < 0$, the compression shock is the line ξ_2 = const. For $A_2 = -2$, the line $\xi_2 = -0.25 A_2$ transforms into a curve which determines its position as a function of the coefficient A_2. As $A_2 \to 0$, the position of the shock front changes only slightly with changes in A_2.

As computations show, for

$$-0.5 < A_2 / A_1 < -0.15 \tag{4.8}$$

the velocity behind the compression shock is supersonic, it becomes subsonic downstream of the line $x = -0.5 A_2 r^2$. Such a flow is mapped in Fig. 10; the corresponding values F_2 and Ψ_2 lie on the curve K_1 in the

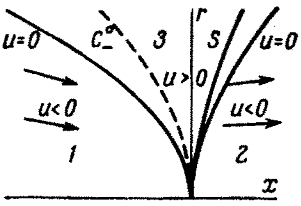

Fig. 10.

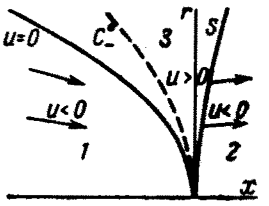

Fig. 11.

right $F\Psi$ half-plane. For $A_2 = -0.15 A_1$ the velocity behind the compression shock equals the critical velocity and its component along the r-axis is positive. Hence, a diminution in the velocity occurs in domain 2. The shock front is simultaneously the line of transition from supersonic to subsonic velocities. The point (0, −2) in the $F\Psi$ plane corresponds thereto and the equation of the shock in the physical plane is written as $x = 0.775\ A_1 r^2$.

In the range of values

$$-0.15 < A_2/A_1 < 0 \tag{4.9}$$

the velocity behind the shock front is subsonic and continues to decrease along the direction to the exhaust part of the nozzle. Such a flow is pictured in Fig. 11; the corresponding values F_2 and Ψ_2 lie in the curve K_1 in the left $F\Psi$ half-plane.

The dependence of $f_3 - f_2$ on A_2 is given on the left half of Fig. 8 for $A_2 < 0$ as constructed taking account of (3.14). As follows from the presented computations, the amplitude of the compression shock achieves its greatest value at $A_2 = -0.0875\ A_1$. Formulas (4.8) and (4.9) show that the particle velocity behind the most intense compression shock for a given value A_1 is less than the speed of sound. For values of A_2 close to the left end of the interval (4.7) the amplitude of the shock front grows slowly as A_2 increases.

5. Construction of the nozzle profiles. Let us elucidate how the nozzle shape changes in the neighborhood of the critical section as a function of the magnitude of the derivative $\partial u/\partial x$ in domain 2. It has already been noted above that the flow to the left of the C_-°-characteristic is conveniently considered to be the same for all the considered gas motions. Hence, the entrance section of all nozzles will be the same, their shape will differ in the region to the right of the C_-°-characteristic.

It should be emphasized that the entire investigation carried out here is of purely local character and refers only to the direct vicinity of the center of the flow. Let us take two streamlines located symmetrically with respect to the x-axis and constructed in conformance with the solution of the Cauchy problem (1.2) as the nozzle walls. Hence, the equation of the channel walls at the point of intersection with the C_+°-characteristic will have a discontinuity in the third derivative, the slope of the wall with respect to the x-axis will change by a jump at the intersection with the shockwave. It is natural that it is impossible to get the needed singularities in constructing the profile of a real nozzle. Discontinuities in the derivatives of the velocity components or in the functions themselves, starting from the center of the channel, will later be reflected from its walls. In a number of cases this reflection may affect the nature of the original gas motion. For example, if the velocity is subsonic behind a shockwave extending downstream, then its reflection from the nozzle walls is generally impossible; it is practically impossible to realize the considered flow with local supersonic zones closed on the axis of symmetry. If the corresponding motion exists, with a transition through the speed of sound, the latter is probably also unrealizable. These remarks should be kept in mind in estimating that magnitude of the deformation of the nozzle profile which accompanies a variation in the constant A_2 which will not lead to violation of the flow in the entrance.

Before turning to the construction of the nozzle profiles, let us find the position of the narrowest section. The direction of the velocity is parallel to the x-axis in this section at points located on the nozzle walls. Hence, the position of the critical section is more simply determined by writing the equation of the line along which $v = 0$. Using (1.5), we obtain its mapping in the $F\Psi$ plane

$$\Psi = 2F\,\frac{F-2}{4-F} \tag{5.1}$$

The curve (5.1) is denoted by V^* in Fig. 2.

The line carrying the null value of the transverse velocity component for an analytic nozzle is obtained by using (1.7); it is concave toward the incident stream

$$x = -\frac{1}{6}A_1 r^2$$

As is seen from Fig. 2, on moving along the integral curves of equation (1.12) we intersect the curve V^* before arriving at the infinitely distant point E only when the branches of these curves starting from the point C are included between the lower branch of the curve K_1 and the line K_2. The line K_2 intersects the curve V^* at the point E. The same argument applies to the curves mapping flows with shockwaves and emerging, as has been shown in Section 2, from the point C above the upper branch of the curve K_1.

It follows from the presented analysis that the line of null transverse particle-velocity components is located to the left of the r-axis and is concave to the incident flow when $0.5 < A_2/A_1 \leqslant 1$.

The flow (3.5) in which the velocity is parallel to the nozzle axis on the r-axis is realized in the domain between the singular $C_{\mp}^{0}$-characteristics for $A_2 = 0.5\,A_1$.

In the range $0.25 \leqslant A_2/A_1 < 0.5$ the flow remains shockless but the narrowest channel section is downstream of its center. In this case the line $v = 0$ is concave to the supersonic part of the nozzle. Its equation for $A_2 = 0.25\,A_1$ will be

$$x = {}^1/_{12}\,A_1 r^2 \tag{5.2}$$

Values of the derivative $\partial u/\partial x$ in the domain 2 of shockless flows with local supersonic zones closed on the axis of symmetry are obtained next. The interval $-2 \leqslant A_2/A_1 < -1$ corresponds to nozzles for which the critical section is located to the left of the origin. The value $A_2 = -A_1$ is obtained when the nozzle has two axes of symmetry and $v = 0$ for $x = 0$. The range $-1 < A_2/A_1 \leqslant -0.5$ corresponds to nozzles whose throat is to the right of the origin but which nevertheless guarantee shock free flow in its vicinity.

As is seen from Fig. 2, discontinuous gas motions are always realized in nozzles for which the narrowest section is located to the right of the origin. Let h denote the half-width of the channel throat. Using formula (5.2) we conclude that the flow through a Laval nozzle becomes discontinuous if the critical section is located at a distance $L > 1/12\ A_1 h^2$ downstream of the point of intersection of the sonic curve with the axis of symmetry. Whether the transition through the speed of sound is realized in the gas motion through such a nozzle or whether the velocity field in domain 2 remains subsonic is of no importance: the flow contains a shockwave. If the distance between the throat and the channel center does not exceed the mentioned limit then the flow remains shockless. As computations show, in the range of values of the constant A_2 prescribed by the inequalities

$$0.16 < A_2/A_1 < 0.25, \qquad -0.5 < A_2/A_1 < -0.325$$

the line $v = 0$ will be more curved than the compression shock. When

$$-0.325 < A_2/A_1 < 0.16$$

the shock front is simultaneously the line at which the values of v change from negative to positive when it is crossed.

Let $r^\times$ denote the deflection of the streamline from the line $r =$ const. The quantity $r^\times$ is found by integrating the differential equation

$$dr^\times / dx = (2m_*)^{-1}\, v\,(x,\ r) \tag{5.3}$$

along the selected line $r =$ const. Substituting the fundamental solution (1.7) subjected to the transformation (3.14) into the right-hand side of the relation (5.3), we find the shape of the entrance of the nozzles under consideration

$$2m_* r^\times = \frac{32}{3} r^3 x + 8rx^2 \tag{5.4}$$

The equality (5.4) determines also the shape of an analytic nozzle downstream of the $C_-^{\,0}$-characteristic. The slope of the walls of non-analytic nozzles remains continuous at intersections with the

C_-°-characteristic while their curvature undergoes a discontinuity. Considering the transformation (3.14) to be performed, we obtain the boundary condition for the integration of equation (5.3) in domain *3*

$$2m_* r^{\times} = -\frac{8}{3} r^5 \quad \text{for } x = -r^2 \tag{5.5}$$

Let us consider the singular flow with $A_2 = 0.5\, A_1$, which is described by (3.5) in the domains between the C_+°-characteristics. After substituting them into (5.3) and satisfying the boundary condition (5.5), we obtain the shape of the nozzle profile as

$$2m_* r^{\times} = -\frac{11}{2} r^5 + 3rx^2 - \frac{1}{6}\frac{1}{r^3} x^4 \tag{5.6}$$

Now, let the flow in domain *3* be given by using the general formulas (3.13) where the constants c and d are connected by (3.15). Integration of (5.3) taking account of the boundary condition (5.5) yields in this case

$$\begin{aligned} 2m_* r^{\times} = \frac{1}{3} \{ & [8(8c^3 - 3c^2 - 3c - 2) + \frac{32}{5}\left(c + \frac{1}{2}\right)(c-1)^2(6c-1)] r^5 + \\ & + 8(8c^3 - 3c - 1) r^3 x + 24c^2 r x^2 - \frac{32}{5}\left(c + \frac{1}{2}\right)(c-1)^{1/2}(x + 6cr^2)(x + cr^2)^{3/2} \} \end{aligned} \tag{5.7}$$

Putting $c = 1$ here, let us again return to (5.4). Selecting $c = -0.5$, we obtain a nozzle in which the discontinuities of the first derivatives of the velocity components are formed both on the C_-°-characteristics and on the C_+°-characteristics. The shape of its profile is given by equation

$$2m_* r^{\times} = -6r^5 - \frac{4}{3} r^3 x + 2rx^2 \tag{5.8}$$

Let K denote the curvature of the nozzle walls. In the considered approximation $K = d^2 r^{\times}/dx^2$.

If the flow through the Laval nozzle is not analytic, then as has been noted above, the curvature of its walls undergoes a discontinuity at the intersection with the C_-°-characteristic. The magnitude of this discontinuity for profiles of all non-analytic nozzles will be

$$[K]_1 = -\frac{3}{8m_*} A_1^2 r < 0$$

because the discontinuity in the derivative $\partial u/\partial x$ is defined uniquely for a given value of A_1 upon passage through the C_-°-characteristic.

Let us consider briefly the behavior of the lines u = const along which the particle velocity, the pressure, the density and the temperature retain constant values. According to (1.7), the lines u = const in the entrance of any nozzle are concave to the free stream. In domain *2* these lines are also concave to the free stream when $A_2 > 0$ but they are concave to the exhaust section of the channel when $A_2 < 0$. Let us consider the behavior of the lines u = const in domain *3*. If the flow through the Laval nozzle is analytic, then the lines u = const are concave to the free stream in this whole domain. In shockless non-analytic gas flows near the C_-°-characteristic the lines carrying the same values of the gas parameters are concave to the exhaust section of the nozzle and concave to the free stream in the vicinity of the C_+°-characteristic. The latter statement follows from the fact that, according to (3.1), the derivative $df/d\xi$ is negative at points of the C_-°-characteristic and the mentioned derivative takes on positive values at points located on the C_+°-characteristics. In limiting non-analytic gas flows when the discontinuities in the first derivatives of the velocity components are formed on both singular characteristics the lines u = const are concave to the exhaust section of the nozzle in all of domain *3*. If the flow through the Laval nozzle is analytic, then the lines u = const in all of this domain are concave to the free stream. In shockless non-analytic flows of a gas, near the C_-°-characteristic the lines bearing constant values of the gas parameters are concave to the exhaust section of the nozzle; and in the vicinity of the C_+°-characteristic these lines are concave to the free stream. The latter statement follows from the fact that according to (3.1), the derivative $df/d\xi$ is negative at points of the C_-°-characteristic and the mentioned derivative takes on positive values at points located on the C_+°-characteristic. In limiting non-analytic gas flow when the discontinuities in the first derivatives of the velocity components are formed on both sides of the singular characteristics the lines u = const are concave to the exhaust section of the nozzle in the whole of domain *3*. An analogous situation holds also in all flows where compression shocks are formed. Actually, the derivative $df/d\xi$ is not only negative along the curves in the ξf plane which map the discontinuous gas motions but it also decreases with the motion from the C_-°-characteristic to the compression shock.

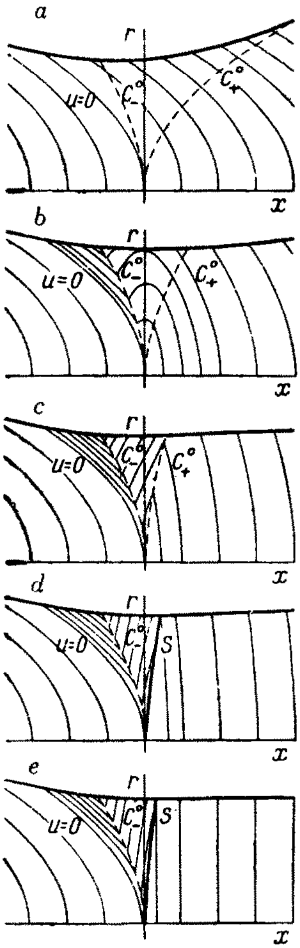

Fig. 12.

The nozzle profiles constructed in conformance with equalities (5.4), (5.6) to (5.8) are presented in Fig. 12. The lines u = const are shown by solid lines there, as are the characteristics C_-^0 arriving at the center of the flow, the characteristics C_+^0 closing the domain *3* of the shockless flows and the compression shocks in the discontinuous flows. All the constructed profiles guarantee supersonic or exactly sonic flow in domain *2*, i.e. $A_2 \geqslant 0$ for all of them. The analytic nozzle with $A_2 = A_1$ is shown in Fig. 12*a*; the nozzle in which the flow (3.5) is realized for $A_2 = 1/2\ A_1$ is mapped in Fig. 12*b*; the nozzle where the gas motion has discontinuities in the first derivatives of the velocity components with respect to the coordinates on both the $C_\mp^0$-characteristics passing through the center is presented in Fig. 12*c*; $A_2 = 1/4\ A_1$ corresponds to it. The flow through the nozzle shown in Fig. 12*d* is obtained for $A_2 = 0.07\ A_1$ and has a compression shock. In the corresponding gas motion with the local supersonic zones being closed on the axis of symmetry, the velocity behind the compression shock equals the speed of sound. Flow through the nozzle presented in Fig. 12*e* corresponds to the value $A_2 = 0$; the gas parameters therein take on critical values in the whole domain *2* behind the shockwave and the velocity behind the compression shock is subsonic in the corresponding motion with local supersonic zones. As has been noted at the beginning of this section, it is impossible to realize such flows in practice.

6. Flows with axial symmetry. For $\nu = 2$, equations (1.1) describe transonic flows with axial symmetry. Let us use them to investigate the character of the passage through sound speed in a channel with a circular cross-section.

For $\nu = 2$ the system of equations (1.1) and the initial data (1.2) are invariant with respect to the same group of similarity transformations as for $\nu = 1$. Hence, the solution of equations (1.1) referred to axisymmetric flows can be sought in the form (1.3). The equation for the function f analogous to (1.4) will have the form

$$(f - 4\xi^2)\frac{d^2 f}{d\xi^2} + \left(\frac{df}{d\xi}\right)^2 + 4\xi\frac{df}{d\xi} - 4f = 0 \tag{6.1}$$

After it has been integrated the function g is determined by means of the formula

$$g = \xi f + {}^1\!/_4 (f - 4\xi^2)\, df/d\xi \tag{6.2}$$

The fundamental solution of (6.1) is written as [8]

$$f = {}^1\!/_4 A^2 + A\xi \tag{6.3}$$

where, as before, the constant A equals the value of the derivative $\partial u/\partial x$ at the point $x = r = 0$. In conformance with formulas (1.3), (6.3) and (6.2), we find

$$u = Ax + {}^1/_4\, A^2 r^2, \qquad v = {}^1/_2\, A^2 x r + {}^1/_{16}\, A^3 r^3 \tag{6.4}$$

For $A = A_1$ the obtained equalities describe the gas motion in the entrance part of the nozzle, for $A = A_2$ they yield the flow in the domain behind the C_+°-characteristic. When $A_1 = A_2$ the whole flow in the vicinity of the center of the channel is determined by (6.4); such a flow is realized in the analytic Laval nozzle. Later we shall consider the constant A_1 to be the same for all the motions under consideration; we shall assume the constant A_2 to be large or equal to zero in the beginning.

The equation of the line of transition through the speed of sound will be

$$x = -{}^1/_4\, A_1 r^2$$

The characteristic curves are found from the solutions of the differential equation

$$(dx\,/\,dr)^2 = A_{1,2}x + {}^1/_4\, A_{1,2}{}^2 r^2$$

The singular $C_{\mp}^{\circ}$-characteristics passing through the center of the nozzle are given by the formulas

$$x = {}^1/_8\, A_1 (1 - \sqrt{5}) r^2 \qquad (C_-^{\circ}\text{-characteristics})$$
$$x = {}^1/_8\, A_2 (1 + \sqrt{5})\, r^2 \qquad (C_+^{\circ}\text{-characteristics})$$

If the flow is devoid of shockwaves then the boundary conditions which must be satisfied when integrating (6.1) in the domain included between the $C_{\mp}^{\circ}$-characteristics are written as

$$\begin{aligned} f = f_1 = {}^1/_8\, A_1^2 (3 - \sqrt{5}) \qquad &\text{for } \xi = \xi_1 = {}^1/_8\, A_1 (1 - \sqrt{5}) \\ f = f_2 = {}^1/_8\, A_2^2 (3 + \sqrt{5}) \qquad &\text{for } \xi = \xi_2 = {}^1/_8\, A_2 (1 + \sqrt{5}) \end{aligned} \tag{6.5}$$

The compression shock in the discontinuous flow will be more curved than the C_+°-characteristic. Hence, only the first of conditions (6.5) for the integration of equation (6.1) in domain *3* remains unchanged in the construction of discontinuous flows. Formula (2.4) and the relation

$$(f_2 - 4\xi_2^2)\, \frac{df_2}{d\xi} + 12\xi_2 f_2 = (f_3 - 4\xi_2^2)\, \frac{df_3}{d\xi} + 12\xi_2 f_3 \tag{6.6}$$

which follows from (2.2) and (6.2) act as the second boundary condition. The shape of the compression shock is given by (2.3).

In order to study the qualitative properties of axisymmetric flows, let us use again the method of the $F\Psi$ phase plane. Using (1.11), let us reduce (6.1) to the form [8]

$$\frac{d\Psi}{dF} = \frac{\Psi^2 + 7\Psi F + 6F^2 - 8\Psi - 4F}{\Psi(4 - F)} \tag{6.7}$$

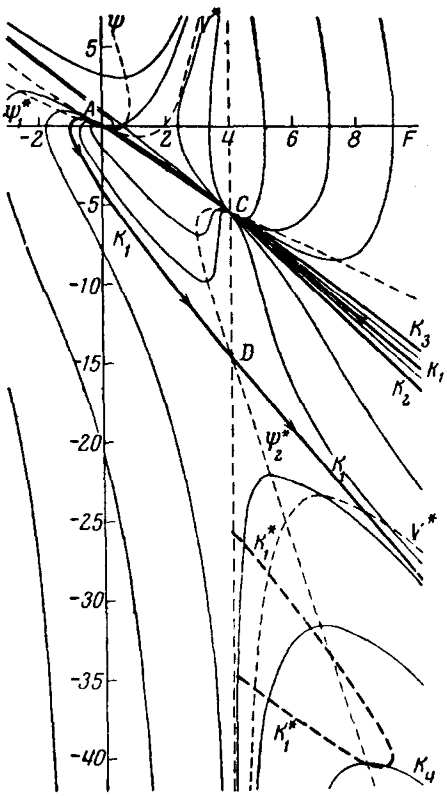

Fig. 13.

The general picture of the field of integral curves of equation (6.7) is shown in Fig. 13 where the notations of the preceding sections are used. The boundary conditions (6.5) which are used in constructing the axisymmetric shockless flows acquire the very same form in the $F\Psi$ plane as for plane-parallel flows

$$F_{1,2} = 4$$

The equality (2.6) and the relation

$$\Psi_2 + \Psi_3 = -40 \tag{6.8}$$

which is obtained after substitution of (1.11) into (6.6) are the boundary conditions on the compression shock.

The mapping of the fundamental solution (6.3) in the $F\Psi$ plane will be called the K_1-curve, just as before, and its equation will be

$$\Psi = -2(1 + F \mp \sqrt{1+F}) \tag{6.9}$$

Motion along the curve K_1 in the direction indicated by the arrow in Fig. 13 corresponds to the flow through an analytic Laval nozzle. As in plane-parallel flows, the domain *3* of shockless non-analytic flows with axial symmetry is mapped in the $F\Psi$ plane by curves emerging from and terminating at the point C. In the vicinity of the inifinitely distant point E they are represented by the expansion

$$\Psi = -2F \pm cF^{1/2} + (4 - {}^3/_2\, c^2) \pm ({}^{15}/_8\, c^2 - 7)\, cF^{-1/2} - {}^2/_3(8 - 14c^2 + 3c^4)\, F^{-1} \pm \ldots$$

where c is an arbitrary constant. Putting $c = 0$, we obtain the equation of the curve K_2

$$\Psi = -2F + 4 - {}^{16}/_3\, F^{-1} + \ldots \tag{6.10}$$

which goes in the forward and reverse directions. One of the branches of all the remaining integral curves of (6.7), which map the shockless non-analytic flows, is included between the curve K_2 and the lower branch of the curve K_1, the other is between this same curve and the upper branch of the curve K_1. In the limiting case, flow having weak discontinuities will be mapped by the curve K_1 traveling in the reverse direction.

Duplicating the discussions presented in Section 1, we find that no discontinuities in the first derivatives of the particle velocity components with respect to the coordinates occur on the $C_+{}^{\circ}$-characteristic of the axisymmetric flows. The expansion of the function $f(\xi)$ in the vicinity of the point $\xi = \xi_2$, corresponding to the $C_+{}^{\circ}$-characteristic also guarantees continuity of the second derivatives of the velocity components on crossing this characteristic.* As before, an exception occurs when the flow between the singular Mach lines is mapped by the curve K_1 traveling in the reverse direction. In the limiting case, the first derivatives of the stream parameters have discontinuities both on the $C_-{}^{\circ}$-characteristics and the $C_+{}^{\circ}$-characteristics.

In the vicinity of the infinitely distant point G the single integral curve of (6.7) passing through it is represented by the expansion

$$\Psi = -\,{}^{3}/_{2}\, F + {}^{2}/_{5} + {}^{4}/_{25}\, F^{-1} + \ldots \tag{6.11}$$

Let us denote the curve (6.11) by K_3, as before.

If the flow between the $C_-{}^{\circ}$-characteristic and the r-axis is mapped in the $F\Psi$ plane by the segment of the curve included between the curve K_3 and the upper branch of the curve K_1 then the flow in the region between the r-axis and the $C_+{}^{\circ}$-characteristic is mapped by the integral curve of (6.7) which is a continuation of that segment and is located below the lower branch of the curve K_1. In such a flow the $C_+{}^{\circ}$-characteristic is simultaneously the limit line. Using the curve K_3 a flow is obtained where the limit line coincides with the r-axis. The integral curves of (6.7) located above the curve K_3 yield examples of flows with limit lines to the left of the r-axis. It is impossible to continue them through the compression shock.

Using (2.6), (6.8) and (6.9), we obtain an equation for the curve K_1*

$$\Psi_3 = -\,(22 + 2F_3 + 2\sqrt{9 - F_3}) \tag{6.12}$$

which relates the quantities F_3 and Ψ_3 characterizing the state of the

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* Let us recall that in plane-parallel flows the $C_+{}^{\circ}$-characteristic bears discontinuities in the second derivatives of the velocity components.

gas ahead of the shockwave. In formula (6.12) $4 < F_3 \leqslant 8$. It is considered that the flow velocity in domain 2 behind the compression shock is larger or equal to the speed of sound. Since only part of the integral curves of (6.7) intersects the curve K_1, not all the flows in which the limit line is the C_+°-characteristic can be realized by the introduction of shockwaves. Exactly as in the plane-parallel flows, in discontinuous flows with axial symmetry the compression shock is generated at the center of the nozzle and then is carried downward to the exhaust section by the stream. The C_-°-characteristic arriving at the nozzle center in all non-analytic gas motions carries discontinuities of the first derivatives of the velocity components with respect to the coordinates.

The behavior of solutions of (6.1) will be qualitatively the same as for the curves presented in Fig. 6. The derivative $df/d\xi$ for all these curves has the value

$$df\,/\,d\xi = -\tfrac{1}{2} A_1 (3 - \sqrt{5}) \tag{6.13}$$

upon approaching the C_-°-characteristic from the right.

Hence, the magnitude of the discontinuity in the derivative $\partial u/\partial x$ on the C_-°-characteristic is expressed uniquely in terms of A_1

$$[\partial u\,/\,\partial x]_1 = -\tfrac{1}{2} A_1 (5 - \sqrt{5}) < 0$$

The equality (3.2) remains valid in shockless flows upon approaching the C_+°-characteristic from the left and right. The boundary conditions (6.5), (6.13) and (3.2) which must be satisfied in integrating (6.1) in domain 3 can be transformed to

$$\begin{array}{lll} f_1 = 4\xi_1^2, & df_1\,/\,d\xi = 2(\sqrt{5} - 1)\,\xi_1 & \text{for } \xi = \xi_1 \\ f_2 = 4\xi_2^2, & df_2\,/\,d\xi = 2(\sqrt{5} - 1)\,\xi_2 & \text{for } \xi = \xi_2 \end{array}$$

These relations show that $df_1/d\xi < 0$ and $df_2/d\xi > 0$, i.e. the function u first decreases in shockless flows and then grows along the lines r = const in domain 3.

In the limiting case of shockless flows, when their mapping in the $F\Psi$ plane is the curve K_1 going in the reverse direction, the solution of (6.1) in domain 3 has the form

$$f = \tfrac{1}{8} A_1^2 (7 - 3\sqrt{5}) - \tfrac{1}{2} A_1 (3 - \sqrt{5})\,\xi \tag{6.14}$$

Satisfying the boundary conditions in the form (6.5), we obtain for the constant A_2

$$A_2 = \tfrac{1}{2} A_1 (7 - 3\sqrt{5})$$

which is analogous to (3.8) for plane-parallel flows. Hence, we have the inequality

$$\tfrac{1}{2}(7-3\sqrt{5}) \leqslant A_2/A_1 \leqslant 1 \tag{6.15}$$

which guarantees the absence of shockwaves in the vicinity of the critical section of a Laval nozzle. The inequalities (3.9) and (6.15) show that for a given value of A_1 lesser values of A_2 are admissible for axisymmetric flows as compared with plane-parallel flows, and this does not lead to the formation of compression shocks.

A curve symmetric with respect to the vertical axis corresponds to the solution (6.10). Using (6.5), we obtain

$$A_2 = \tfrac{1}{2}A_1(3-\sqrt{5})$$

The integral curves of (6.1) above the curve symmetric with respect to the f-axis are obtained for

$$\tfrac{1}{2}(3-\sqrt{5}) < A_2/A_1 < 1$$

The integral curves included between this curve and the line (6.14) are found for

$$\tfrac{1}{2}(7-3\sqrt{5}) < A_2/A_1 < \tfrac{1}{2}(3-\sqrt{5})$$

The correspondence between the curves in the $F\Psi$ plane and the integral curves of (6.1) for axisymmetric flows is exactly the same as for the plane-parallel gas motions examined earlier.

Flows with shockwaves behind which the velocity is supersonic and continues to increase toward the exhaust part of the nozzle correspond to the range of values

$$0 < A_2/A_1 < \tfrac{1}{2}(7-3\sqrt{5}) \tag{6.16}$$

For $A_2 = 0$ the gas parameters in domain 2 take on the critical values and the ξ-axis corresponds to uniform flow behind the compression shock in Fig. 6. In this case the simple equalities

$$f_3 = 8\xi_2^2, \qquad df_3/d\xi = -24\xi_2$$

are the boundary conditions which must be satisfied on the right end of the interval when integrating (6.1). The derivative $df/d\xi$ continuously decreases during motion along the integral curves in the ξf plane in the direction from the $C_-^{\,0}$-characteristic to the compression shock. Shockwave formation in nozzles with circular cross-section is accompanied, as before, by deceleration in the domain behind the singular $C_-^{\,0}$-characteristic more rapid than according to a linear law. In the entrance to the Laval nozzle the flow remains unchanged, the formation of a shock front does not lead to its destruction. A comparison of inequalities (3.10) and (6.16) shows that weaker compression shocks originate in axisymmetric than in plane-parallel flows for the same values of A_1.

The dependence of the quantities ξ_2 and $f_3 - f_2$ on A_2 has the same character as that shown in Figs. 7 and 8. It is simplest to obtain this dependence by direct integration of (6.1) from a given point $\xi = \xi_1$ to a certain point $\xi = \xi_2$ where the boundary conditions (2.4) and (6.6) are satisfied on the shock front.

Let us consider axisymmetric flows in which local supersonic zones which are closed on the axis are formed in the vicinity of the channel throat. As in the plane-parallel gas motions, their parameters in domains *1* and *3* agree with the parameters of the corresponding flows where the transition through the speed of sound is performed. In domain *1* the flow is described by (6.4) with $A = A_1$; in domain *2* by the same formulas with $A = A_2$, where $A_1 > 0$ and $A_2 < 0$. The transition through the speed of sound when moving along the streamline from domain *3* into domain *2* is performed either at points of the curve $x = -1/4\, A_2 r^2$ or on the compression shock.

In shockless flows the equation of the C_+°-characteristic is written as

$$x = {}^1/_8\, A_2 (1 - \sqrt{5})\, r^2$$

There the boundary condition for the integration of (6.1) in the domain included between the singular Mach lines is

$$f = f_2 = {}^1/_8\, A_2^2 (3 - \sqrt{5}) \qquad \text{for } \xi = \xi_2 = {}^1/_8\, A_2 (1 - \sqrt{5}) \tag{6.17}$$

The flow with local supersonic zones which corresponds to gas motion with a transition through the speed of sound in an analytic Laval nozzle in the $F\Psi$ plane is mapped by the curve K_1 traveling in the forward direction from the point A to the point C with a subsequent jump from the point C to the point D. As before, the C_-°-characteristic is devoid of any singularities while the first derivatives of the velocity components with respect to the coordinates undergo discontinuities on the C_+°-characteristic [10].

Satisfying the boundary condition (6.17), we obtain the equation

$$A_2 = -{}^1/_2 A_1 (3 + \sqrt{5})$$

which is the minimum value of the constant A_2 for a given A_1. Using the last relation we find

$$[\partial u / \partial x]_2 = -\tfrac{1}{2} A_1 (5 + \sqrt{5})$$

In all the remaining flows with local supersonic zones, the C_-°-characteristic arriving at the center of the nozzle carries discontinuities of the first derivatives of the velocity vector components. The equality (6.13) remains valid for passage through it. In shockless flows the C_+°-characteristic also carries discontinuities in the first derivatives of the velocity components with respect to the coordinates. The magnitude of this discontinuity is determined by the formula

$$[\partial u / \partial x]_2 = \tfrac{1}{2} A_2 (5 - \sqrt{5}) < 0$$

Only in a flow mapped by the curve K_1 traveling in the reverse direction does the C_+°-characteristic not carry any singularities. In domains *3* and *2* the solution of (6.1) is described by (6.14). Hence, we have the largest value of A_2 for which shockwaves do not originate in a flow with local supersonic zones closed on the x-axis

$$A_2 = -\tfrac{1}{2} A_1 (3 - \sqrt{5})$$

The range of values of the constant A_2 corresponding to shockless flows is therefore determined by the inequalities

$$-\tfrac{1}{2} (3 + \sqrt{5}) \leqslant A_2 / A_1 \leqslant -\tfrac{1}{2} (3 - \sqrt{5}) \tag{6.18}$$

A comparison of inequalities (4.5) and (6.18) shows that larger values of A_2 are admissible for axisymmetric flows with local supersonic zones as compared with analogous plane-parallel flows, and these larger values do not lead to the origination of compression shocks. The inequalities (6.18) supplement relations (6.15) which refer to flows with the transition through the speed of sound in the vicinity of the critical section of the channel. A flow symmetric with respect to the r-axis corresponds to the solution (6.10) of equation (6.7); hence, $A_2 = -A_1$ for it.

The study of axisymmetric flows with local supersonic zones having shockwaves leads to the very same deductions which were made in the study of analogous plane-parallel gas motions. In the majority of flows with shockwaves the gas parameters in domains *1* and *3* agree with the parameters of the corresponding motions where the transition through the speed of sound is performed in the vicinity of the critical section of the nozzle. In such flows the state of the medium ahead of the compression shock is mapped either by points of the upper branch

$$\Psi_3 = -(22 + 2F_3 - 2\sqrt{9 - F_3})$$

of the curve K_1^*, where $4 < F_3 \leqslant 9$ or by points of the lower branch

(6.12) of this same curve, where $F_s < F_3 \leqslant 9$. The value F_s of the quantity F is obtained at the intersection of the curve $K_1{}^*$ with the integral curve K_4 of (6.7) which maps the flow with critical values of the parameters in domain 2. Only such integral curves of (6.7) as do not yield flows with a transition through the critical velocity for any intersection with the lower branch of the curve $K_1{}^*$, pass below the curve K_4 in the $F\Psi$ plane.

The range of values of A_2 to which discontinuous motions correspond is defined by the inequalities

$$-{}^1/_2(3-\sqrt{5}) < A_2/A_1 < 0 \tag{6.19}$$

The character of the dependence of ξ_2 and $f_3 - f_2$ on A_2 for discontinuous axisymmetric flows with local supersonic zones is qualitatively the same as for analogous plane-parallel flows. For example, the particle velocity behind the compression shock most intense for a given value of A_1 is less than the critical velocity. A comparison of inequalities (4.7) and (6.19) discloses that the compression shocks which originate in axisymmetric flows have less intensity than in plane-parallel gas motions.

In conclusion, let us analyze how the profile of a nozzle with a circular cross-section changes as a function of the magnitude of the derivative $\partial u/\partial x$ in domain 2. Using (6.2), we find the mapping V^* of the line $v = 0$ in the $F\Psi$ plane; as before, it is given by (5.1). Duplicating the reasoning presented in Section 5, we find that the line $v = 0$ is located to the left of the r-axis and is concave to the free stream

$$^1/_2(3-\sqrt{5}) < A_2/A_1 \leqslant 1$$

The line of zero transverse particle velocity for an analytic nozzle with $A_2 = A_1$ has the form $x = -1/8\ A_1 r^2$.

For $A_2 = 1/2\ A_1(3 - \sqrt{5})$ a flow is realized in the neighborhood of the nozzle throat in which the velocity vector is parallel to the nozzle axis along the r-axis.

In the range of values of A_2 prescribed by the relations

$$^1/_2(7-3\sqrt{5}) \leqslant A_2/A_1 < {}^1/_2(3-\sqrt{5})$$

the flow remains shockless but the line $v = 0$ is concave to the exhaust section of the nozzle. Its equation for $A_2 = 1/2\ A_1(7 - 3\sqrt{5})$ will be

$$x = {}^1/_{16}\, A_1(3-\sqrt{5})\, r^2 \tag{6.20}$$

The values of the constant A_2 for shockless flows with local supersonic zones closed on the axis will be in the following range:

$$-\tfrac{1}{2}(3+\sqrt{5}) \leqslant A_2/A_1 < -1$$

where the line $v = 0$ is concave toward the free stream.

The range

$$-1 < A_2/A_1 \leqslant -\tfrac{1}{2}(3-\sqrt{5})$$

corresponds to nozzles for which the line of the zero transverse velocity component is concave to the exhaust section.

The value $A_2 = -A_1$ corresponds to flow through a nozzle having two axes of symmetry. There $v = 0$ for $x = 0$.

Discontinuous gas motions are realized only in nozzles for which the critical section is located to the right of the origin. Using (6.20) we find that the flow through a Laval nozzle remains continuous if the distance L between the throat and the center of the channel does not exceed the quantity $1/16\ A_1(3-\sqrt{5})h^2$ where h denotes the radius of the throat. If

$$L > \tfrac{1}{16} A_1 (3-\sqrt{5}) h^2$$

the flow becomes discontinuous independently of whether the transition through the speed of sound is accomplished in the gas motion or the velocity field in domain *2* is subsonic.

In constructing solutions of (6.1) in domains *3* and *2* it is convenient to put

$$A_1 = 2(1+\sqrt{5}) \tag{6.21}$$

Then, as in the plane-parallel flows, the value of the function f will be 4 upon crossing the C_-°-characteristic at $\xi = -1$. The equality (6.21) is equivalent to the transformation to the new variables

$$\xi \to \frac{\xi}{2(1+\sqrt{5})/A_1}, \qquad f \to \frac{f}{8(3+\sqrt{5})/A_1^2}$$

Integration of (5.3) in whose right-hand side the fundamental solution (6.4) has been substituted, results when (6.21) is taken into account in the formula

$$2m_* r^{\nu} = 4(2+\sqrt{5})\, r^3 x + 2(3+\sqrt{5})\, r x^2 \tag{6.22}$$

The described relation yields the entrance section of all the nozzles under consideration; it also determines the shape of an analytic nozzle downstream of the C_-°-characteristic. Using (6.22), we find the boundary condition for integration of (5.3) in domain *3*

$$2m_* r^{\varkappa} = -2(1+\sqrt{5})\, r^5 \qquad \text{for } x = -r^2$$

Let us analyze a flow with discontinuities of the first derivatives of the particle velocity components on both sides of the characteristics passing through the center of the nozzle. The equation of its profile is written as

$$2m_* r^{\varkappa} = -4\sqrt{5}r^5 + 4(2-\sqrt{5})\, r^3 x + 2(3-\sqrt{5})\, r x^2$$

The magnitude of the discontinuity in K_1, the wall curvature, of all non-analytic nozzles with circular cross-section at the intersection with the $C_-^{\,0}$-characteristic is given by the equality

$$[K]_1 = \frac{1}{8m_*}(5-3\sqrt{5})\, A_1^2 r < 0$$

The behavior of the lines u = const along which the gas parameters remain constant will qualitatively be the same in flows through axisymmetric nozzles as in plane-parallel motions.

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